MODAL DENSITIES OF SPHERICAL SHELLS

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This paper presents formulas of modal density of nonshallow spherical homogeneous and sandwich shells according to classical bending theory, and in viscoelastic medium. The influences of transverse, longitudinal, and rotary inertia terms and transverse shear deformation on modal density are examined numerically. Numerical results show that, for some values of certain parameters, radius-thickness ratio and transverse shear rigidity are of considerable importance.

INTRODUCTION

Solutions for response of shells subjected to external excitation can be derived easily on the basis of modal analysis theoretically, but numerical calculation of these solutions, especially to high frequency excitation, may become extremely laborious. New methods are needed to simplify calculations.

Some information is available on the concept of modal density of structural elements [1, 2], which is defined as the number of modes (i.e., the number of natural frequencies) within a unit frequency band; this concept is useful for the calculation of the response of a structure to high frequency excitation [3].

Detailed calculations of modal densities for shallow structural elements have been made by Bolotin [2], Wilkinson [4], and Erickson [5] during the past few years, but not for nonshallow elements. Still, for the estimation of the response level of a nonshallow spherical shell in which the assumptions used in shallow structural elements are not applicable, modal density may become an important physical parameter to broadband acoustic excitation.

The primary purpose of the present paper is to obtain the modal densities of nonshallow spherical shells. Another purpose is to investigate the influence of rotary, longitudinal inertia terms and transverse shear deformation on modal densities in the vibration in the high frequency range.

Modal density of a closed isotropic homogeneous spherical shell is derived first according to classical bending theory; modal density of an open sphere and a shallow cap is then examined in relation to that of the closed sphere. Also, the modal density of an isotropic sandwich sphere is shown. In each case modal densities are presented for a nondamping state and for a linear viscous damping state. Numerical results for homogeneous and sandwich closed shells are presented in graphic form and the character of modal density is examined.

SYMBOLS

a = radius of spherical shell (Fig. 1)

d = parameter of transverse shear rigidity of isotropic homogeneous shell, $d = 1/\bar{D}\alpha$

d_s = parameter of transverse shear rigidity of isotropic sandwich core, $d_s = 1/\bar{D}_Q\alpha_s$

D = bending rigidity, $Eh^3/12(1-\nu^2)$

$\bar{D}$ = transverse shear deformation rigidity of isotropic homogeneous element, $5/6Gh = 5/6[Eh/2(1+\nu)]$

$\bar{D}_Q$ = transverse shear deformation rigidity of isotropic sandwich core

E, E_s = Young's modulus

g = acceleration of gravity

h = shell thickness for homogeneous shell or distance between surface sheets for sandwich (Fig. 1)

k = $12a^2/h^2$

k_s = $4a^2/h^2$

$\bar{m}$ = weight of shell element per unit area

q = $a^2/k\bar{D}$

q_s = $a^2/k_s\bar{D}_Q$

p_o = some given frequency

p = circular frequency

$P_m^n(\cos\phi)$ = Legendre function of first kind

$Q_m^n(\cos\phi)$ = Legendre function of second kind

t = time

t_s = thickness of surface sheet of sandwich element (Fig. 1)

α = parameter of extensional rigidity, $(1-\nu^2)/Eh$

α_s = parameter of extensional rigidity of sandwich element, $(1-\nu^2)/2E_s t_s$

γ = characterized elastic modulus of viscoelastic medium

δ = viscoelastic damping parameter

ν = Poisson's ratio

ω = angular velocity

Ω = modified frequency, $\Omega^2 = \alpha a^2 (\bar{m}/g)\omega^2$

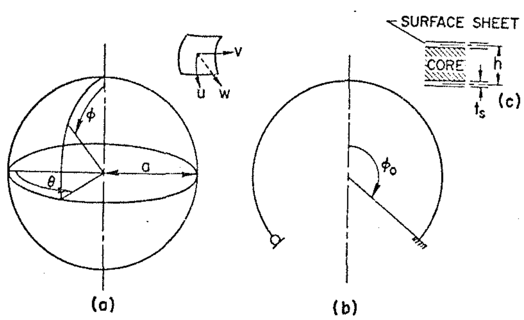


Fig. 1. (a) Coordinate system of sphere; (b) Open spherical shell; (c) Sandwich section

Closed Isotropic Homogeneous Spherical Shell

From the compatibility conditions and kinetic equations where transverse, longitudinal, rotary inertia terms and transverse shear deformation are taken into consideration, the governing equation with respect to transverse deflection w of a nonshallow spherical shell due to classical bending theory can be given in the differential equation of 6th order as follows (see Ap. A),

$$\{H_2H_2H_0 - (1-\nu^2)dH_2H_2 + k(1-\nu^2)H_2 + [-(2+d)H_2H_0 + kH_2 - 3(1+\nu)k]\,\alpha a^2 L + [(2d+1)H_2 - k]\,\alpha^2 a^4 LL - \alpha^3 a^6 d\, LLL\}(w) = 0 \qquad (1)$$

where

$$H_0 = \frac{\partial^2}{\partial\phi^2} + \cot\phi\,\frac{\partial}{\partial\phi} + \frac{1}{\sin^2\phi}\,\frac{\partial^2}{\partial\theta^2} \qquad H_2 = H_0 + 2$$

$$L = \frac{\bar{m}}{g}\,\frac{\partial^2}{\partial t^2}$$

Homogeneous solutions of (1) can be given in the following form [6].

$$w = \sum_{n=0}^{\infty}\sum_{i=1}^{3}\left[C_i^n P_{\mu i}^n(\cos\phi) + D_i^n Q_{\mu i}^n(\cos\phi)\right]e^{j\omega t}\cos n\theta \qquad (2)$$

where

$$\mu_i = -\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_i} \qquad j = \sqrt{-1}$$

and λ_i = functions of angular velocity and other shell parameters, given as roots of a cubic equation. For modified frequency Ω in rather large range ($\Omega \geq 1$), λ_i takes real value (large positive, negative small absolute value or negative of large absolute value). Therefore, mode shape of deflection (2) for fixed n and frequency parameter Ω is constituted by three fundamental mode shapes, one with amplitude decreasing rapidly from the boundary, another with a long wavelength in the ϕ direction and the last one with a short wavelength in the ϕ direction. The latter determines the mode shape of the overall shell vibration as shown in reference [6]. As $-\lambda_i = \mu_i(\mu_i + 1)$, it can be considered that μ may take positive integer for negative λ of large absolute value. That is, when λ is negative and $|\lambda|$ is very large the fractional order μ can be replaced by nearest integer with the sufficient accuracy. It can be assumed that in a spherical shell, the influence of boundary conditions on the deflection is negligible in the vibration of high frequency from references [7] and [8]. Then neglecting the boundary conditions and using the Legendre's bipolynomial, we can set the deflection of Eq. (1) as follows,

$$w = \sum_{\substack{n=0 \\ (m \geq n)}}^{\infty} \sum_{m=n}^{\infty} E_m^n e^{j\omega t} \begin{Bmatrix} P_m^n(\cos\phi) \\ \\ Q_m^n(\cos\phi) \end{Bmatrix} \begin{Bmatrix} \sin n\theta \\ \\ \cos n\theta \end{Bmatrix} \quad (3)$$

Assumption of expression (3) for the deflection seems to be quite satisfactory, because (3) is a series of orthogonal functions and any arbitrary function can be expanded in the series of (3).

Functions $P_m^n(\cos\phi)$, $Q_m^n(\cos\phi)$ are trigonometric expansions summed through $\cos m\phi$ or $\sin m\phi$. Suffix m indicates the mth order harmonic mode shape in ϕ direction. By substitution of (3) into (1), the following equation can be deduced.

$$X^3 + AX^2 + BX + C = 0 \quad (4)$$

where

$X = m(m+1)$
A, B, C = functions of modified frequency Ω and other shell parameters, given later.

Equation (4) may be regarded as a relation between only m and Ω, but there is the limitation $m \geq n$. The number of frequencies N(p) smaller than a given frequency $p = p_0$ corresponds to the number of lattice points of (m, n) in the domain S in Fig. 2(a) and may be determined approximately from the area S for large p_0, as has been indicated in many references.

Generally it may be said that the difficulty in the calculation of modal densities exists in obtaining the area S, which is given in a double integral with curved boundaries. Here the expression (3) of deflection simplifies the calculation of this area, as is indicated in Fig. 2(a).

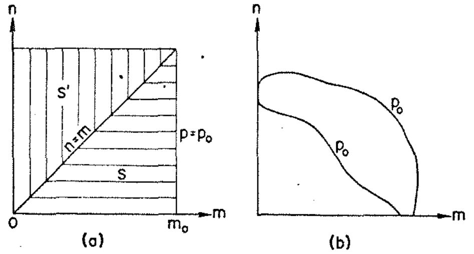


Fig. 2. (a) Wave number bounded by a given frequency for spherical shell; (b) Constant frequency curves given by Equation (4)

Note that the number of frequencies that correspond to the complimental area S' in Fig. 2(a) has to be taken into consideration by the exchange of coordinates ϕ and θ in Eq. (3) (i.e., rotation of coordinate axes ϕ and θ). Therefore, in the calculation of modal density, the area $S' + S = 2S$ should be used;

$$N(p) \approx 2S = m^2 = \frac{1}{2}\left(2X + 1 - \sqrt{4X+1}\right) \quad X \geq 0 \quad (5)$$

and with the definition of modal density

$$\begin{aligned} M(p) \approx 2\frac{\partial S}{\partial p} &= 4\pi\,\frac{\partial S}{\partial \omega} \\ &= 2\pi\sqrt{\alpha a^2 \frac{\bar{m}}{g}}\left(\frac{-1}{4X+1} + 1\right)\frac{\partial X}{\partial \Omega} \\ &= 2\pi\sqrt{\alpha a^2 \frac{\bar{m}}{g}}\left(\frac{1}{4X+1} - 1\right) \\ &\quad \frac{X^2(\partial A/\partial\Omega) + X(\partial B/\partial\Omega) + \partial C/\partial\Omega}{3X^2 + 2AX + B} \end{aligned} \quad (6)$$

The relation of coefficients shows that Eq. (4) provides two positive and one negative roots for rather large values of Ω (for instance, $\Omega \geq 1.0$); then the relation between a given frequency p_o and (m, n) can be shown ideally in Fig. 2(b). Note that we use the larger area for the calculation of the modal density. That is, the largest root of Eq. (4) for a given frequency should be used in Eq. (6), which must be rewritten as:

$$M_o(p) \approx 2\pi\sqrt{\alpha a^2 \frac{\bar{m}}{g}}\left(\frac{1}{4X_o+1} - 1\right) \left[\frac{(\partial A/\partial\Omega)X_o^2 + (\partial B/\partial\Omega)X_o + \partial C/\partial\Omega}{3X_o^2 + 2AX_o + B}\right]_{\Omega = \Omega_o} \quad (6a)$$

where X_o equals the maximum root (positive) of (4) for Ω_o.

$$\left.\begin{aligned} A &= -4 + (1-\nu^2)d - (2+d)\Omega^2 \\ B &= 4 - 4(1-\nu^2)d + (1-\nu^2)k - k\Omega^2 + 2(2+d)\Omega^2 \\ &\quad + (1+2d)\Omega^4 \\ C &= 4(1-\nu^2)d - 2(1-\nu^2)k - (1+3\nu)k\Omega^2 \\ &\quad + [k - 2(2d+1)]\Omega^4 - d\Omega^6 \end{aligned}\right\} \quad (7)$$

If the influence of transverse shear deformation is negligible, we can set $d = 0$ in Eq. (7). If we wish to neglect the terms due to rotary inertia, the governing equation presented in reference [6] must be used. Equations (4) and (6a) remain valid, however, and the coefficients are given as follows:

$$\left.\begin{aligned} A &= -4 - \Omega^2 \\ B &= 4 + (1-\nu^2)k - k\Omega^2 + 2\Omega^2 \\ C &= -2(1-\nu^2)k - (1+3\nu)k\Omega^2 + k\Omega^4 \end{aligned}\right\} \quad (8)$$

If the only transverse inertia term is taken into consideration, the coefficients become

$$\left.\begin{aligned} A &= -4 \\ B &= 4 - k\Omega^2 + (1 - \nu^2)k \\ C &= -2(1 - \nu^2)k + 2k\Omega^2 \end{aligned}\right\} \tag{9}$$

Open Shell (Nonshallow Domain)

Spherical shells are often used in an open form as shown in Fig. 1(b), and Eq. (6a) is not applicable for an open sphere. An approximate calculation of modal density, as follows, is proposed.

As we are concerned with the response in the high frequency band, the mode index number m in (3) takes on very large values. If we consider the nature of high-order Legendre bipolynomials with respect to $\cos\phi$, the following expression may be applicable for the deflection with sufficient accuracy,

$$w = \sum_{n=0}^{\infty} \sum_{m'=n}^{\infty} F_{m'}^{n} e^{j\omega t} \begin{Bmatrix} P_{m'}^{n}(\cos\phi) \\ Q_{m'}^{n}(\cos\phi) \end{Bmatrix} \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \tag{10}$$

where

m' = integer and equal to $(\pi/\phi_0)m$
m = mode index number for an open sphere (not necessarily an integer)
ϕ_0 = open angle [i.e., Fig. 1(b)]

Here the possible set of (m', n) must be taken in the range $0 < m'$, $n \le m_0$ in spite of the restriction of $m' \ge n$ in Eq. (10), because in (10) the mode shapes are given by exchange of coordinate axes exclusive, similar to (3). Substituting (10) into (1), we can calculate modal density by the procedure described earlier:

$$M\phi_0(p) \approx \left(\frac{\phi_0}{\pi}\right)^2 M_0(p) \tag{11}$$

where $M_0(p)$ is given in (6a). The governing range of (11) for ϕ_0 will be shown below.

Shallow Shell

Previous papers presented the modal density of a spherical cap, though in the Cartesian coordinates. The author will show only the relation between the shallow shell and the aforementioned shell in circular coordinates.

With the approximation of $\sin\phi \to \phi$, $\cot\phi \to 1/\phi$, the governing Eq. (1) can be presented as

$$\begin{aligned} \{ I_2 I_2 I_0 &- (1 - \nu^2) d I_2 I_2 + k(1 - \nu^2) I_2 \\ &+ [-(2 + d) I_2 I_0 + k I_2 - 3(1 + \nu)k]\, \alpha a^2 L \\ &+ [(2d + 1) I_2 - k]\, \alpha^2 a^4 LL - \alpha^3 a^6 d LLL \} w = 0 \end{aligned} \tag{12}$$

where

$$I_0 = \frac{\partial^2}{\partial\phi^2} + \frac{1}{\phi}\frac{\partial}{\partial\phi} + \frac{1}{\phi^2}\frac{\partial^2}{\partial\theta^2} \qquad I_2 = I_0 + 2$$

or

$$(I_0 + \lambda_1)(I_0 + \lambda_2)(I_0 + \lambda_3) w = 0 \tag{12a}$$

Then we can set the deflection as

$$w = \sum_{n=0}^{\infty} \sum_{i=1}^{3} e^{j\omega t} J_n\left(\sqrt{\lambda_i}\,\phi\right) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \tag{13}$$

In Bessel's differential equation with extremely large index n, eigenvalue $\sqrt{\lambda}$ can be given approximately in the following form [1].

$$\lambda = m^2\pi^2$$

Therefore, substitution of (13) into (12) yields the relation between p_0 and m, which is similar to (4), and modal density is given as

$$M_s(p) \approx 2\pi\left(\frac{1}{\pi^2}\right)\frac{\partial X}{\partial\omega} \tag{14}$$

where $X = m^2\pi^2$ and the cubic equation with respect to X becomes the same as (4).

The open shell modal density proposed in (11) may be rewritten for rather large m as

$$M\phi_0(p) \approx 2\pi\left(\frac{\phi_0}{\pi}\right)^2 \frac{\partial X}{\partial\omega}$$

In comparison with this and (14), expression (11) is valid in such a domain of open angle ϕ_0 as

$$1 \le \phi_0 \le \pi \tag{15}$$

Sandwich Shell

Thin sandwich panels with a small cell shape usually are used in the formation of complex shapes such as nonshallow spheres. Then the assumption remains valid that h/a is sufficiently small in comparison with unity, and the thickness of surface sheets of sandwich panel becomes so much smaller than the other size parameter that the bending stiffness of these face sheets is negligible. Moreover, it may be presumed that the sandwich shell has the same nature in any circumferential direction, that is, isotropic in ϕ and θ directions.

The governing equation in this case is (see Ap. B):

$$\Big\{ H_2H_2H_0 - (1-\nu^2)d_s H_2H_2 + (1-\nu^2)k_s H_2 + \Big[-\Big(\frac{4}{3}+d_s\Big)H_2H_0 + k_sH_2 - 3(1+\nu)k_s\Big]\alpha_s a^2 L + \Big[\frac{1}{3}(1-2d_s)H_2 - k_s\Big]\alpha_s^2 a^4 LL - \frac{1}{3}\alpha_s^3 a^6 d_s LLL\Big\} w = 0 \tag{16}$$

where

$$\alpha_s = \frac{1-\nu^2}{2E_s t_s} \qquad k_s = \frac{4a^2}{h^2} \qquad d_s = \frac{1}{\alpha_s \bar{D}_Q}$$

Setting the deflection as in (3) and substituting into (16), we can obtain the coefficients as follows:

$$A = -4 + (1-\nu^2)d_s - \Big(\frac{4}{3}+d_s\Big)\Omega^2$$

$$B = 4 + k_s(1-\nu^2) - k_s\Omega^2 - 4(1-\nu^2)d_s + 2\Big(\frac{4}{3}+d_s\Big)\Omega^2 + \frac{1}{3}(1-2d_s)\Omega^4$$

$$C = -2(1-\nu^2)k_s + 4(1-\nu^2)d_s + 2k_s\Omega^2 - 3(1+\nu)k_s\Omega^2 + \Big[k_s - \frac{2}{3}(1-2d_s)\Big]\Omega^4 - \frac{1}{3}d_s\Omega^6 \tag{17}$$

Eqs. (4) and (6a) are valid in this case, and then Eqs. (11) and (14) are still valid for a nonshallow open shell and a shallow shell, respectively.

Spherical Shell in a Viscoelastic Medium

When a shell lies in a viscoelastic medium, the differential operator with respect to time t takes the following form:

$$L = \frac{\bar{m}}{g}\frac{\partial^2}{\partial t^2} + \delta\frac{\partial}{\partial t} + \gamma \tag{18}$$

The square of the modified frequency can be given as (see Ap. C):

$$\Omega_0^2 = \Omega^2 + \alpha a^2\Big[\frac{\delta^2}{4(\bar{m}/g)} - \gamma\Big] \tag{19}$$

where

$$\Omega_0^2 = \alpha a^2(\bar{m}/g)\omega_0^2 \quad \text{nondamping}$$

$$\Omega^2 = \alpha a^2(\bar{m}/g)\omega^2 \quad \text{value of this case}$$

Modal density can be calculated by substitution of (19) into (4) and (6a) and differentiation with respect to ω.

$$M_v(p) \approx 4\pi\frac{\partial S}{\partial \omega} = 4\pi\frac{\partial S}{\partial \omega_0}\frac{\partial \omega_0}{\partial \omega} = M_0(p)\frac{\sqrt{\Omega_0^2 - \alpha a^2[\delta^2/4(\bar{m}/g)] - \gamma}}{\Omega_0} \tag{20}$$

for

$$\Omega_0^2 - \alpha a^2\Big[\frac{\delta^2}{4(\bar{m}/g)} - \gamma\Big] \geq 0$$

and

$$M_v(p) = 0 \tag{21}$$

for

$$\Omega_0^2 - \alpha a^2\Big[\frac{\delta^2}{4(\bar{m}/g)} - \gamma\Big] \geq 0$$

Numerical Calculation

If modal density (6a) has any singular point, it must be in such a case that roots of Eq. (4) satisfy

$$3X^2 + 2AX + B = 0$$

simultaneously. This takes place for the equal roots of Eq. (4), which are negative for a value of $0.8 < \Omega < 1.0$ from the relation of coefficients and roots of (4), or more clearly in numerical calculation [6]. The remaining root is positive and as in the calculation of modal density, maximum root of Eq. (4) is required for any Ω. Thus, there is no singular point in (6a).

Eigenvalues Ω that correspond to low frequency modes are assumed to be in the domain $0.8 < \Omega < 1.0$ [6], [7], and response in such a low frequency range is calculated more accurately by the classical modal analysis method than by the modal density method. Therefore, numerical results will be shown for $\Omega > 1$.

The variations in modal density of an isotropic homogeneous shell (6a) with modified frequency Ω are shown in Figs. 3(a) and (b) for a/h = 50 and 100, respectively. In the calculation the following values are used:

$$\bar{D} = \frac{5}{6}Gh \qquad G = \frac{E}{2(1+\nu)} \qquad d = \frac{12}{5(1-\nu)}$$

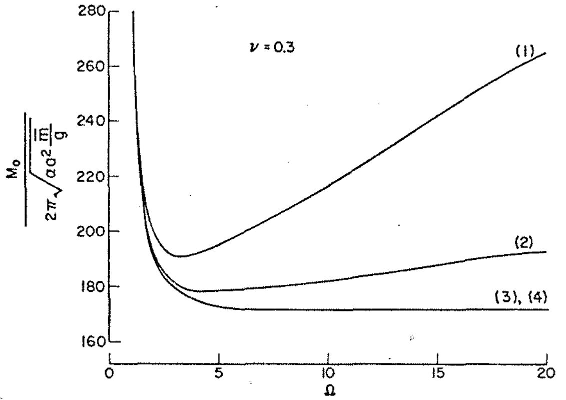
and four curves are considered.

i. (transverse + longitudinal + rotary) inertia terms + transverse shear deformation

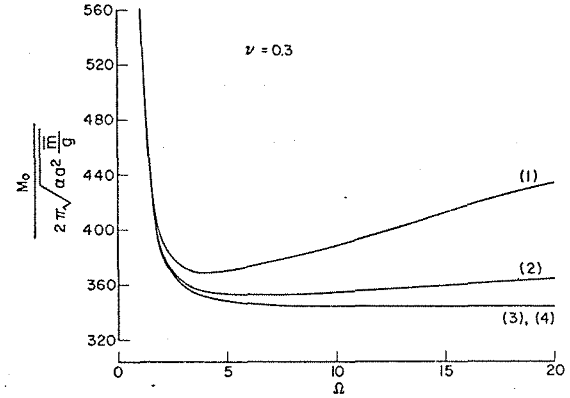
ii. (transverse + longitudinal + rotary) inertia terms

iii. (transverse + longitudinal) inertia terms

iv. transverse inertia terms



(a) $\frac{a}{h} = 50$



(b) $\frac{a}{h} = 100$

Fig. 3. Modal density of an isotropic homogeneous closed spherical shell

From Fig. 3, we note:

1. When rotary inertia and transverse shear deformation are included in the kinematic equation, modal density increases with frequency (see curves i and ii). This indicates that these terms may not be negligible in the analysis of response in the high frequency band.

2. The difference between curve (i) and the others indicates the considerable influence of transverse shear deformation on modal density. Variation of a/h appears to influence slightly the ratio of increase of curve i with frequency.

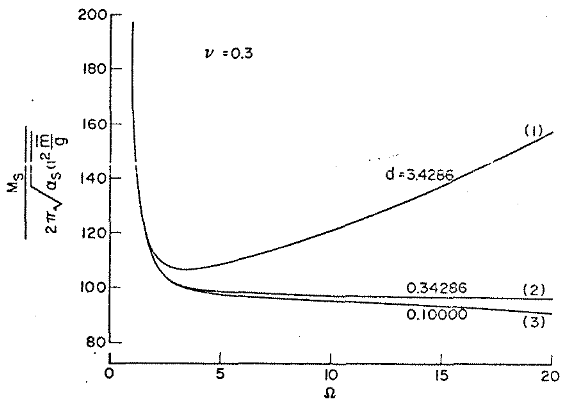
3. There is no difference between curves iii and iv; that is, modal density is scarcely affected by longitudinal inertia terms in spite of rather high frequency range. Note that modal density as considered here relates to transverse deflection. Curve iii and iv become flat rapidly and keep constant value.

4. In modal density radius-thickness ratio a/h seems to be the most important parameter as may be seen by the change in level in the curves from Figs. 3(a) and (b).

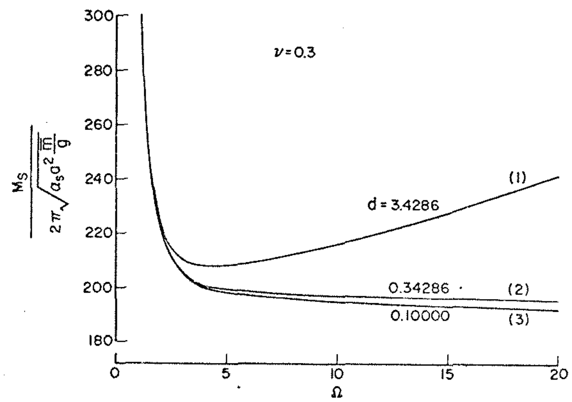
Modal density of isotropic sandwich shells are given in Fig. 4(a) and (b) for a/h = 50 and 100, respectively, with variations in shear deformation rigidity. With the assumption that $\bar{D}_Q = 5/6[(E_s h)/2(1 + \nu)]$, numerical computation is considered for three cases ($\nu = 0.3$):

1. $\frac{2t_s}{h} = 1.0 \rightarrow d_s = \frac{12}{5(1 - \nu)}$
2. $\frac{2t_s}{h} = 0.1 \rightarrow d_s = \frac{1.2}{5(1 - \nu)}$
3. $\frac{2t_s}{h} = \frac{0.5(1 - \nu)}{12} \rightarrow d_s = 0.1$

Terms of transverse, longitudinal, and rotary inertia and transverse shear deformation are taken into consideration in these three cases.



(a) $\frac{a}{h} = 50$



(b) $\frac{a}{h} = 100$

Fig. 4. Modal density of an Isotropic sandwich closed shperical shell

1. Curve (1) represents the model density for extremely large t_s, and in curves (2) and (3) shear deformation rigidity of sandwich core becomes relatively large. Results show the considerable effect of transverse shear deformation in modal density.

2. When the ratio t_s/h decreases, its influence on the curves becomes rapidly slight, and for a rather small ratio, modal density decreases with increased frequency. We are also able to determine the ratio t_s/h to make modal density constant.

3. The decrease and increase ratio of curves with frequency becomes small with increase of a/h in all cases, and this point is a little different from the homogeneous shell.

4. In modal density, the ratio a/h seems to be the most important parameter.

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APPENDIX A

On the basis of classical bending theory, kinetic equations of spherical shell elements are deduced from reference [9] as follows:

$$(\sin\phi\, N_{\theta\phi})^o + N'_\theta + \cos\phi\, N_{\theta\phi} - \sin\phi\, Q_\theta - a\sin\phi\, L(v + ak_2\beta_\theta) = 0 \qquad \text{(A1a)}$$

$$(\sin\phi\, N_\phi)^o + N'_{\theta\phi} - \cos\phi\, N_\theta - \sin\phi\, Q_\phi - a\sin\phi\, L(u + ak_2\beta_\phi) = 0 \qquad \text{(A1b)}$$

$$(\sin\phi\, Q_\phi)^o + Q'_\theta + \sin\phi\,(N_\theta + N_\phi) - a\sin\phi\, L(w) = 0 \qquad \text{(A1c)}$$

$$(\sin\phi\, M_\phi)^o + M'_{\theta\phi} - \cos\phi\, M_\theta + a\sin\phi\, Q_\phi - \frac{a^2}{k} a\sin\phi\, L\left(\frac{2}{a}u + k_\gamma\beta_\phi\right) = 0 \qquad \text{(A1d)}$$

$$(\sin\phi\, M_{\theta\phi})^o + M'_\theta + \cos\phi\, M_{\theta\phi} + a\sin\phi\, Q_\theta - \frac{a^2}{k} a\sin\phi\, L\left(\frac{2}{a}v + k_\gamma\beta_\theta\right) = 0 \qquad \text{(A1e)}$$

where

$$a\beta_\phi = w^o + u - \frac{a}{\bar{D}} Q_\phi$$

$$a\beta_\theta = \frac{1}{\sin\phi} w' + v - \frac{1}{\bar{D}} Q_\theta$$

$$k_\phi = \frac{1}{a}\beta^o_\phi \qquad k_\theta = \frac{1}{a}\left(\frac{1}{\sin\phi}\beta'_\theta - \beta_\phi \cot\phi\right)$$

$$k_{\theta\phi} = \frac{1}{a}\left(\beta^o_\theta + \frac{1}{\sin\phi}\beta'_\phi - \cot\phi\, \beta_\theta\right)$$

$$k_2 = \frac{2}{k} \qquad k_\gamma = 1 + \frac{9}{5k} \qquad k = \frac{12a^2}{h^2}$$

$$o = \frac{\partial}{\partial\phi} \qquad ' = \frac{\partial}{\partial\theta} \qquad L = \frac{\bar{m}}{g}\frac{\partial^2}{\partial t^2}$$

and see reference [6].

Approximating as $k_\gamma - k_2 \approx 1$, we can deduce one differential equation from (A1a) through (A1c) as follows:

$$\begin{aligned} &H_2H_0(w) - \alpha a^2 LH_0(w) + 2(1+\nu)k\left(1 + \frac{1}{k} + qL\right)w \\ &+ \alpha k a^2\left(1 + \frac{1}{k} + qL\right)L(w) - (1+\nu)k\left(1 + \frac{1}{k} + qL\right)\chi \\ &- 2\alpha a^2 L(\chi) - \frac{a^2}{\bar{D}} H_1 L(w) + \frac{1}{\bar{D}}\frac{Eh}{1-\nu} H_1(\chi) \\ &- \frac{2}{\bar{D}}\frac{Eh}{1-\nu} H_1(w) = 0 \end{aligned} \qquad \text{(A2)}$$

where

$$H_1 = H_0 + (1-\nu)$$

$$q = \frac{a^2}{k\bar{D}}$$

$$\chi = \frac{1}{\sin\phi}(\sin\phi\, u)^o + \frac{v'}{\sin\phi}$$

Compatibility condition can be given in the following equation from strain-displacement relations and constitutive equations:

$$\begin{aligned} &H_2(\chi) - (1+\nu)H_2(w) - \alpha a^2 L(w) - \alpha a^2 L(\chi) \\ &- 2(1+\nu)qL(\chi) - \alpha a^2 k_2 L(\chi) + 2\alpha a^2 qLL(w) \\ &+ 4(1+\nu)qL(w) - \alpha a^2 k_2 LH_0(w) = 0 \end{aligned} \qquad \text{(A3)}$$

Eliminating χ from (A2) and (A3), we can get finally,

$$\Big\{ H_2H_2H_0 - (1-\nu^2)dH_2H_2 + \Big[(1+\nu)(1-\nu^2) + k\Big(1+\frac{1}{k}+qL\Big)\Big(1-\nu^2+\alpha a^2L\Big)\Big]H_2$$

$$- (2+k_2+d)\alpha a^2 H_2H_0 L$$

$$- \Big\{\Big[<1-\nu> - 8<1+\nu>^2 \frac{d}{k}\Big]d + (1+\nu)(4+k_2 - d<3+k_2-\nu k_2> + 2qL<1 - d>)\Big\}\alpha a^2 LH_2$$

$$- \Big[\Big(1+\nu\Big)\Big(<1+\nu>d + k<1+\frac{1}{k}+qL>\Big)(3+k_2+4<1+\nu>dk_2) + k\alpha a^2\Big(1+\frac{1}{k}+qL\Big)\Big(1+k_2+<1+\nu>dk_2\Big)L\Big]$$

$$\alpha a^2 L + [1+d+(2d+\nu<1+d>)k_2]\alpha^2a^4H_2LL$$

$$- [4+(1+\nu)d + 7(1+\nu)dk_2 + 2\nu k_2 - (1-\nu^2)d^2k_2 + 4qL]\alpha^2a^4LL\Big\}(w) = 0$$

where usually $a/h \gg 1$ and then $k \gg 1$. Neglecting $1/k$ in comparison with unity and using $\bar{D} = 5/6\,Gh$ and $G = (E)/2(1+\nu)$, we can get the governing equation with respect to deflection w of an isptropic homogeneous shell.

$$\Big\{H_2H_2H_0 - (1-\nu^2)dH_2H_2 + (1-\nu^2)kH_2 + [-(2+d)H_2H_0 + kH_2 - 3(1+\nu)k]\alpha a^2L + [(2d+1)H_2 - k]\alpha^2a^4LL - \alpha^3a^6dLLL\Big\}$$

$$w = 0 \tag{A4}$$

where

$$d = \frac{12}{5(1-\nu)} \equiv 0(1) \ll k$$

If we intend to neglect transverse shear deformation only, we can set $\bar{D}$ equal to infinity, that is, $d = 0$ in (A4).

For the calculation of displacement components u and v, solutions of another differential equation that is independent from Eq. (A4) must be added to those of (A4), and for this equation w becomes equal to zero [6].

APPENDIX B

For sandwich shell, if we set

$$D = \frac{E_s t_s h^2}{2(1-\nu^2)} \qquad Eh = 2E_s t_s \qquad \alpha_s = \frac{1-\nu^2}{2E_s t_s}$$

$$d_s = \frac{1}{\bar{D}_Q}\frac{1-\nu^2}{2E_s t_s} \qquad k_s = \frac{4a^2}{h^2}$$

and rewrite inertia terms in (A1d) and (A1e) as

$$-\frac{a^2}{3k_s} a \sin\phi L\Big(\frac{2}{a}u + k_\gamma \beta_\phi\Big)$$

$$-\frac{a^2}{3k_s} a \sin\phi L\Big(\frac{2}{a}v + k_\gamma \beta_\theta\Big)$$

kinetic equations and compatibility equation of Ap. A are valid. Finally we can deduce the governing equation as,

$$\Big(H_2H_2H_0 - (1-\nu^2)d_sH_2H_2 + \Big[(1+\nu)(1-\nu_2) + k_s\Big(1+\frac{1}{k}+\frac{q_s}{3}L\Big)(1-\nu^2+d_sa^2L)\Big]H_2$$

$$-\Big(\frac{4}{3}+\frac{2}{k_s}+d_s\Big)\alpha_s a^2 H_2H_0 L$$

$$-\Big\{\Big[(1-\nu) - 8(1+\nu)^2\frac{d_s}{k_s}\Big]d_s + (1+\nu)\Big[2+\frac{2}{k_s} - d_s\Big(3+<1-\nu>\frac{2}{k_s}\Big) + 2q_s\Big(\frac{1}{3}-d_s\Big)L\Big]\Big\}\alpha_s a^2H_2L$$

$$-\Big\{(1+\nu)\Big[(1+\nu)d_s + k_s\Big(1+\frac{1}{k_s}+\frac{q_s}{3}L\Big)\Big]\Big[3+\frac{2}{k_s}+4(1+\nu)d_s\frac{2}{k_s}\Big] + k_s\alpha_s a^2\Big[1+\frac{1}{k_s}+\frac{1}{3}q_sL\Big]\Big[1+\frac{2}{k_s}+<1+\nu>d_s\frac{2}{k_s}\Big]L\Big\}\alpha_s a^2L$$

$$+\alpha_s^2a^4\Big[\frac{1}{3}-d_s\Big]\Big\{1+\frac{2}{k_s}+2[1+\nu]d_s\frac{1}{k_s}\Big\}LLH_2$$

$$+\Big[1-\nu\Big]\alpha_s^2a^4d_s\Big\{1+\frac{2}{k_s}+<1+\nu>\frac{2}{k_s}d_s\Big\}LL\Big)$$

$$w = 0$$

Here we assume that transverse shear rigidity $\bar{D}_Q$ takes such values as

$$\bar{D}_Q = \frac{5}{6}\frac{E_s h}{2(1+\nu)}$$

then d_s becomes

$$d_s = \frac{12}{5(1-\nu)}\left(\frac{2t_s}{h}\right) \le \frac{24}{5(1-\nu)} \equiv 0(1)$$

If $k_s = 4a^2/h^2$ is extremely large in comparison with unity, we can obtain the governing equation approximately but with sufficient accuracy:

$$\left\{H_2H_2H_0 - (1-\nu^2)d_sH_2H_2 + (1-\nu^2)k_sH_2 + \left[-\left(\frac{4}{3}+d_s\right)H_2H_0 + k_2H_2 - 3(1+\nu)k_s\right]\alpha_s a^2 L + \left[\frac{1}{3}(1-2d_s)H_2 - k_s\right]\alpha_s^2 a^4 LL - \frac{1}{3}\alpha_s^3 a^6 d_s LLL\right\} w = 0 \qquad \text{(B1)}$$

APPENDIX C

For the problem of free vibration in a visco-elastic medium, the differential operator L( ) with respect to time t in kinetic equation (A1a) through (A1e) becomes

$$L = \frac{\overline{m}}{g}\frac{\partial^2}{\partial t^2} + \delta\frac{\partial}{\partial t} + \gamma \qquad \text{(C1)}$$

We may treat more simple kinetic equations where only transverse and longitudinal inertia are included instead of (A1a) through (A1e) without loss of generality. In the governing equation of this state, we can set

$$w = we^{j\omega t} \qquad \omega = \text{complex value}$$

and obtain the following equation.

$$\left\{H_2H_2H_0 + \Omega^2 H_2H_0 + \left[k(1-\nu^2) - k\Omega^2\right]H_2 + 3(1+\nu)k\Omega^2 - k(1-\nu)\Omega^4\right\} w_0 = 0 \qquad \text{(C2)}$$

where

$$\Omega^2 = \frac{a^2}{Eh}\left(1-\nu^2\right)\left(\frac{\overline{m}}{g}\omega^2 - j\delta\omega - \gamma\right)$$

$$= A(\omega_0^2 - \tau^2) + B\tau - C + j(2\omega_0\tau A - B\omega_0)$$

$$= \Omega_1^2 + j\Omega_2^2 \qquad \text{(C3)}$$

$$A = \frac{(1-\nu^2)\overline{m}a^2}{Ehg}$$

$$B = \frac{(1-\nu^2)a^2}{Eh}\delta$$

$$C = \frac{(1-\nu^2)a^2}{Eh}\cdot\gamma$$

$$\omega = \omega_0 + j\tau$$

Substitution of (C3) into (C2) yields:

$$\left[H_2H_2H_0 + \Omega_1^2H_2H_0 + k(1-\nu^2-\Omega_1^2)H_2 + 3(1+\nu)k\Omega_1^2 - k(1-\nu^2)(\Omega_1^4 - \Omega_2^4)\right] w = 0 \qquad \text{(C4)}$$

$$\left[\Omega_2^2H_2H_0 - k\Omega_2^2H_2 + 3(1+\nu)k\Omega_2^2 - 2k(1-\nu^2)\Omega_1^2\Omega_2^2\right] w = 0 \qquad \text{(C5)}$$

Then

i. If $\Omega_2^2 = 0$, Eq. (C5) is satisfied identically.

ii. If $\Omega_2^2 \neq 0$, deflection w must satisfy (C4) and (C5) simultaneously.

This state takes place only when Ω_1 and Ω_2 take some constant value identically without consideration of boundary conditions. We can conclude that the state $\Omega_2^2 \neq 0$ is not generally valid.

As an eigenvalue problem, we set $\Omega_2 = 0$ and $\tau = B/2A$. Then

$$\omega_0^2 = \frac{1}{A}\left(\Omega_1^2 - \frac{B^2}{4A} + C\right) \qquad \text{(C6)}$$

and Ω_1 = eigenvalue of nondamping problem

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DISCUSSION

Mr. Pearson (NASA Ames Research Ctr.): What is the significance of the minimum point of the curves at a frequency parameter of approximately three?

Mr. Kunieda: Only curve one has such a low point. The reason is that only above that frequency does the transverse shear term become significant.